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Integrable quadratic classical Hamiltonians on $so(4)$ and $so(3, 1)$

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Abstract
We investigate a special class of quadratic Hamiltonians on $so(4)$ and $so(3, 1)$ and describe Hamiltonians that have additional polynomial integrals. One of the main results is a new integrable case with an integral of sixth degree.

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1. Introduction

In this paper we consider the following family of Poisson brackets:

$$\{M_i, M_j\} = \varepsilon_{ijk} M_k, \qquad \{M_i, \gamma_j\} = \varepsilon_{ijk} \gamma_k, \qquad \{\gamma_i, \gamma_j\} = \kappa \varepsilon_{ijk} M_k. \tag{1.1}$$

Here M_i and γ_i are components of three-dimensional vectors $\mathbf{M}$ and $\boldsymbol{\Gamma}$, ε_{ijk} is the totally skew-symmetric tensor, κ is a parameter. It is well known that any linear Poisson bracket is defined by an appropriate Lie algebra. The cases $\kappa = 0$, $\kappa > 0$ and $\kappa < 0$ correspond to the Lie algebras $e(3)$, $so(4)$ and $so(3, 1)$.

Bracket (1.1) has the two Casimir functions:

$$J_1 = (\mathbf{M}, \boldsymbol{\Gamma}), \qquad J_2 = \kappa |\mathbf{M}|^2 + |\boldsymbol{\Gamma}|^2,$$

where $(\cdot, \cdot)$ stands for the standard dot product in $\mathbb{R}^3$. Hence, for the Liouville integrability of the equations of motion only one additional integral functionally independent of the Hamiltonian and the Casimir functions is necessary.

There are two examples of classical problems with the $e(3)$-bracket $\kappa = 0$ and a Hamiltonian of the above form:

(i) the Euler–Poinsot model describing the motion of a rigid body around a fixed point under gravity (with $H = (\mathbf{M}, A\mathbf{M}) + (Q, \boldsymbol{\Gamma})$), and

(ii) the Kirchhoff model describing the motion of a rigid body in ideal fluid with

$$H = (\mathbf{M}, A\mathbf{M}) + (\mathbf{M}, B\mathbf{\Gamma}) + (\mathbf{\Gamma}, C\mathbf{\Gamma}). \tag{1.2}$$

Homogeneous Hamiltonians (1.2) have numerous applications for $\kappa \neq 0$ each with extra conditions on matrices A, B, C. If κ is positive then they include

(A) the Poincaré model describing free motion (without gravity) of a three-dimensional body with an ellipsoidal cavity filled with ideal fluid around a fixed point,
(B) a model describing the motion of a four-dimensional rigid body around a fixed point (in flat space as the Poincaré model),
(C) a Hamiltonian describing two-spin interactions [1].

For positive *and* negative κ the Hamiltonians (1.2) include as well cases of motion of a three-dimensional rigid body in a space of constant curvature κ either (D) freely (see [2–4]), or (E) in ideal fluid (generalized Kirchhoff model). In [3] a model with several ellipsoidal cavities that is more general than the Poincaré model (A) is given, where the restrictions on matrices A, B, C in (1.2) are less strict.

By adding linear terms the above models can be generalized. For example, for model (A) linear terms could be used to describe the motion of bodies with multiply connected cavities, for model (B) to add gyrostatic momentum to the rigid body to become a gyrostat[3] and for model (C) linear terms are related to a constant magnetic field.

It should also be noted that any integrable quadratic homogeneous Hamiltonian defines a left-invariant metric on $SO(4)$ or $SO(3, 1)$ with integrable geodesic flows and that such a Hamiltonian provides an interesting model for quantization (see, for example [5]).

Many papers have been devoted to search and investigation of integrable Hamiltonians (1.2). The case where the Hamiltonian (1.2) has a linear additional integral of motion has been investigated by Poincaré [2]. There are two classical integrable cases, one found by Frahm-Schottky and one by Steklov, where the additional integral of motion is of second degree. It was proved in [6] that any Hamiltonian (1.2) possessing an additional second degree integral is equivalent to one of these two cases. In 1986 Adler–van Moerbeke [7] and Reyman–Semenov-tian Shansky [8] independently found a Hamiltonian of the form (1.2) with fourth degree additional integral.

For the Frahm-Schottky, Steklov and Adler–van Moerbeke–Reyman–Semenov-Tian-Shansky cases all matrices A, B and C are diagonal. This special subclass of 'diagonal' Hamiltonians (1.2) was investigated by many authors but no new integrable cases were found. Probably only these three integrable cases exist among 'diagonal' Hamiltonians.

In the paper [9] the first integrable 'non-diagonal' Hamiltonian (1.2) with $\kappa = 0$ (i.e. a Hamiltonian of the Kirchhoff type describing the motion of a rigid body in ideal fluid) was found. This Hamiltonian has a fourth degree additional integral. Generalizations of this Hamiltonian that include the Kowalewski Hamiltonian have been given in [10, 11] and a different generalization to the case $\kappa \neq 0$ was reported in [12].

In paper [13] the Hamiltonian has been rewritten in the form

$$H = (\mathbf{M}, A\mathbf{M}) + (\mathbf{b}, \mathbf{M} \times \Gamma), \tag{1.3}$$

where A is a constant symmetric matrix, $\mathbf{b} \neq 0$ is a constant vector and $\times$ stands for the skew product. It turns out that this class is very rich in integrable cases. In paper [5] all Hamiltonians (1.3) with a quartic additional integral were described. Moreover, it was mentioned in

[3] A mechanical system is called gyrostat if a rigid body is connected to other bodies in such a way that their relative motion does not change the distribution of mass of the system.

[5, 14, 15] that the general Sklyanin brackets [16] for the XXX-magnetic model lead to some integrable Hamiltonians of the same kind.

The goal of our paper is a systematic investigation of integrable real Hamiltonians (1.3) and their inhomogeneous generalizations

$$H = (\mathbf{M}, A\mathbf{M}) + (\mathbf{b}, \mathbf{M} \times \mathbf{\Gamma}) + (\mathbf{k}, \mathbf{M}) + (\mathbf{n}, \mathbf{\Gamma}), \tag{1.4}$$

where $\mathbf{k}$ and $\mathbf{n}$ are constant vectors. Note that the transformations

$$\mathbf{M} \to T\mathbf{M}, \qquad \mathbf{\Gamma} \to T\mathbf{\Gamma} \tag{1.5}$$

for any constant orthogonal matrix T preserve brackets (1.1) and the form of Hamiltonian (1.4). Using transformations (1.5), one can reduce any (real) Hamiltonian (1.3) to

$$H = a_1 M_1^2 + a_2 M_2^2 + a_3 M_3^2 + a_4 M_1 M_3 + a_5 M_2 M_3 + M_1\gamma_2 - M_2\gamma_1. \tag{1.6}$$

In this canonical form the vector $\mathbf{b}$ is normalized to $(0, 0, 1)$. The alternative idea of bringing matrix A to the diagonal form lead to overwhelming computational complexity.

Using different integrability tests, we find in sections 2 and 3 a list of integrable Hamiltonians (1.4). For each Hamiltonian found in these sections we have verified that the corresponding additional integral is functionally independent of the Hamiltonian and the Casimir functions J_1 and J_2.

All computations there have been made without any serious computer algebra packets and can be easily verified. In section 4 some statements about the completeness of this list are reported. To obtain the results of section 4, very cumbersome computations have been done by the specialized computer algebra package Crack. It is designed to solve overdetermined polynomial differential and algebraic systems with an emphasis on extremely large problems. In [18] an overview of the package and examples for its use in the classification of integrable systems are given.

It is marvelous that all known Hamiltonians (1.3) possessing additional polynomial integrals have also some **linear** partial integrals. These linear partial integrals turn out to be factors of the polynomial integrals for all homogeneous Hamiltonians considered in this paper. For this reason we start our study with a simple integrability test based on the existence of linear partial integrals (see subsection 2.1). An interesting subclass of one-parametric families of Hamiltonians (1.3) arises there.

In the next subsection 2.2 we apply the Kowalewski–Lyapunov test, which is well known in the Painlevé analysis, to find all possibly integrable families from this subclass. Because of a continuous parameter in the Hamiltonian this test becomes extremely efficient. The main result of this consideration is a new integrable Hamiltonian on $so(3,1)$ with an additional sixth degree integral.

In section 3 we are dealing with inhomogeneous Hamiltonians (1.4). Fixing an integrable homogeneous Hamiltonian as a quadratic part of (1.4), one can easily find all possible linear parts that preserve the integrability. We present a list of such linear parts for Hamiltonians found in section 2.

As we claim in section 4, there are no Hamiltonians of the form (1.3) having additional integrals of degrees from 1 to 8 other than examples described in section 2. We have also verified that we found in section 3 *all* Hamiltonians (1.4) having additional integrals of degrees from 1 to 6.

In this paper we present integrable Hamiltonians and the corresponding additional integrals in a general vector form, which is invariant with respect to the orthogonal transformations. This notation allows us to reduce the length of expressions for the Hamiltonians and the integrals of motion and bring them to a more elegant form. For computations with vectorial expressions an extra code was written.

2. Homogeneous integrable cases

2.1. Linear partial integrals

In this section we describe all Hamiltonians of the form (1.3) having linear partial integrals

$$P = (\mathbf{u}, \mathbf{M}) + (\mathbf{v}, \mathbf{\Gamma}), \tag{2.7}$$

where $\mathbf{u}$ and $\mathbf{v}$ are constant vectors. By definition of linear partial integral, the following relation

$$\{H, P\} = [(\mathbf{p}, \mathbf{M}) + (\mathbf{q}, \mathbf{\Gamma})] \cdot P \tag{2.8}$$

has to be hold for some constant vectors $\mathbf{p}$ and $\mathbf{q}$. This implies that the corresponding equations of motion preserve the constraint $P = 0$.

Relation (2.8) is equivalent to a system of bi-linear algebraic equations for coefficients of the Hamiltonian and components of vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{p}$ and $\mathbf{q}$. Below we present the result of our investigation of this system.

One can check that there are two different possibilities, either case 1: $\mathbf{v} = \mathbf{q} = 0$, or case 2: vector $\mathbf{q}$ is equal to vector $\mathbf{b}$ from formula (1.3).

Case 1. Calculations show that any Hamiltonian having a linear partial integral with $\mathbf{v} = \mathbf{q} = 0$ belongs to a class of 'vectorial' Hamiltonians of the form

$$H = c_1(\mathbf{a}, \mathbf{b})|\mathbf{M}|^2 + c_2(\mathbf{a}, \mathbf{M})(\mathbf{b}, \mathbf{M}) + (\mathbf{b}, \mathbf{M} \times \mathbf{\Gamma}), \tag{2.9}$$

where $\mathbf{b}$ and $\mathbf{a}$ are constant vectors, c_i are constant scalars. In this case $\mathbf{u} = \mathbf{b}$ and $\mathbf{p} = c_2 \mathbf{a} \times \mathbf{b}$. In other words, for Hamiltonian (2.9) we have

$$\{H, (\mathbf{b}, \mathbf{M})\} = c_2(\mathbf{a} \times \mathbf{b}, \mathbf{M}) \cdot (\mathbf{b}, \mathbf{M}). \tag{2.10}$$

In the next subsection we will present four different integrable Hamiltonians of the form (2.9) that possess additional integrals of degrees 1, 3, 4 and 6.

Case 2. In the case $\mathbf{q} = \mathbf{b}$ the following conditions

$$(\mathbf{b}, \mathbf{v}) = (\mathbf{b}, \mathbf{u}) = 0, \qquad \mathbf{p} = \xi \mathbf{b},$$

where ξ is a scalar, have to be fulfilled.

Case 2a. If the vectors $\mathbf{v}$ and $\mathbf{u}$ are not parallel, then without loss of generality we may assume that $\mathbf{b} = \mathbf{u} \times \mathbf{v}$. It turns out that in this case $\xi = 0$ and $\mathbf{u}$ and $\mathbf{v}$ are arbitrary vectors such that $(\mathbf{u}, \mathbf{v}) = 0$. The Hamiltonian is given by

$$H = \frac{1}{2}|\mathbf{u}|^2|\mathbf{M}|^2 + \frac{1}{2}(\mathbf{u}, \mathbf{M})^2 - \frac{\kappa}{2}(\mathbf{v}, \mathbf{M})^2 + (\mathbf{u} \times \mathbf{v}, \mathbf{M} \times \mathbf{\Gamma}). \tag{2.11}$$

The partial integral $P = (\mathbf{u}, \mathbf{M}) + (\mathbf{v}, \mathbf{\Gamma})$ satisfies the relation

$$\{H, P\} = (\mathbf{u} \times \mathbf{v}, \mathbf{\Gamma}) \cdot P.$$

This Hamiltonian has the following additional integral of fourth degree:

$$I = (P|\mathbf{M}|^2 - 2(\mathbf{v}, \mathbf{M})(\mathbf{M}, \mathbf{\Gamma})) \cdot P. \tag{2.12}$$

This integrable case was found in a different non-vector form in [9, 12]. A Lax operator is presented in [19].

Case 2b. The other possibility is that the vectors $\mathbf{v}$ and $\mathbf{u}$ are parallel:

$$(\mathbf{b}, \mathbf{v}) = 0, \qquad \mathbf{p} = \xi \mathbf{b}, \qquad \mathbf{u} = \eta \mathbf{v}.$$

It turns out that in this case $\xi = \eta$ and $\kappa = \eta^2$. We see that a real linear integral exists only for the $so(4)$-version $\kappa > 0$ of bracket (1.1). The Hamiltonian is given by the formula

$$H = -2\eta(\mathbf{v}, \mathbf{M})(\mathbf{z}, \mathbf{M}) + (\mathbf{b}, \mathbf{M} \times \mathbf{\Gamma}), \qquad \mathbf{b} = \mathbf{v} \times \mathbf{z} \tag{2.13}$$

where $\mathbf{v}$ and $\mathbf{z}$ are arbitrary constant vectors. The partial integral $P = (\mathbf{v}, \eta\mathbf{M} + \mathbf{\Gamma})$ satisfies

$$\{H, P\} = (\mathbf{b}, \eta\mathbf{M} + \mathbf{\Gamma}) \cdot P.$$

This Hamiltonian has the following additional integral of fourth degree (cf [16, 14]):

$$I = (\mathbf{z}, (\eta\mathbf{M} - \mathbf{\Gamma})|\mathbf{M}|^2 + 2\mathbf{M}(\mathbf{M}, \mathbf{\Gamma})) \cdot P. \tag{2.14}$$

The eigenvalues α_i of the matrix A from (1.2) satisfy the following relations:

$$\alpha_3 = 0, \qquad \alpha_1\alpha_2 = -\eta^2|\mathbf{b}|^2.$$

Diagonalizing the matrix A, we get the possible canonical form of the Hamiltonian (2.13)

$$H = \eta\left(cM_1^2 - \frac{1}{c}M_2^2\right) + M_1\gamma_2 - M_2\gamma_1.$$

Although it is real only for the $so(4)$-bracket, after the renormalization $c = \frac{\bar{c}}{\eta}$ of the arbitrary parameter c we get the Hamiltonian (see [5])

$$H = \bar{c}M_1^2 - \frac{\kappa}{\bar{c}}M_2^2 + M_1\gamma_2 - M_2\gamma_1,$$

which is real for any bracket (1.1). In particular, if $\kappa = 0$, we have a new integrable case on $e(3)$ with a fourth degree integral.

2.2. Kowalewski–Lyapunov test and a new integrable case

In this section we show that the class of Hamiltonians (2.9) contains a number of integrable cases. The first example of this kind was found in [5]:

Example 1. Consider the $so(3, 1)$-version $\kappa < 0$ of bracket (1.1). The Hamiltonian

$$H = (\mathbf{a}, \mathbf{b})|\mathbf{M}|^2 - (\mathbf{a}, \mathbf{M})(\mathbf{b}, \mathbf{M}) + (\mathbf{b}, \mathbf{M} \times \mathbf{\Gamma}), \tag{2.15}$$

where the vector $\mathbf{b}$ is arbitrary and the length of the vector $\mathbf{a} = (a_1, a_2, a_3)$ is related to the Poisson bracket parameter κ by

$$a_1^2 + a_2^2 + a_3^2 = -\kappa, \tag{2.16}$$

possesses the additional quartic integral

$$I = (\mathbf{b}, \mathbf{M})^2[2(\mathbf{a}, \mathbf{M} \times \mathbf{\Gamma}) - (\mathbf{a}, \mathbf{M})^2 - \kappa|\mathbf{M}|^2 + |\mathbf{\Gamma}|^2].$$

Recently in paper [15] the following integrable case has been found:

Example 2. The Hamiltonian

$$H = (\mathbf{a}, \mathbf{b})|\mathbf{M}|^2 - 2(\mathbf{a}, \mathbf{M})(\mathbf{b}, \mathbf{M}) + (\mathbf{b}, \mathbf{M} \times \mathbf{\Gamma}), \tag{2.17}$$

has under condition (2.16) the additional cubic integral

$$I = (\mathbf{b}, \mathbf{M})[2(\mathbf{a}, \mathbf{M} \times \mathbf{\Gamma}) - \kappa|\mathbf{M}|^2 + |\mathbf{\Gamma}|^2].$$

In the two examples mentioned above, constraint (2.16) is necessary for integrability. We are going to find all integrable Hamiltonians similar to (2.15) and (2.17). To do that we apply the Kowalewski–Lyapunov test to the class of Hamiltonians (2.9) assuming that the additional condition (2.16) is valid.

Suppose we have a dynamical system

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} = \mathbf{F}(\mathbf{X}), \qquad \mathbf{X} = (x_1, \ldots, x_N), \qquad \mathbf{F} = (f_1, \ldots f_N), \tag{2.18}$$

where f_i are homogeneous quadratic polynomials of $\mathbf{X}$. Solutions of the form

$$\mathbf{X}_0 = \frac{\mathbf{K}}{t} \tag{2.19}$$

for system (2.18) with $\mathbf{K}$ being a constant vector are called *Kowalewski solutions*. Substituting (2.19) into (2.18), one obtains a system of algebraic equations for possible vectors $\mathbf{K}$.

The linearization $\mathbf{X} = \mathbf{X_0} + \varepsilon\boldsymbol{\Psi}$ of system (2.18) on a Kowalewski solution $\mathbf{X_0}$ obeys

$$\frac{\mathrm{d}\boldsymbol{\Psi}}{\mathrm{d}t} = \frac{1}{t} S(\boldsymbol{\Psi}), \tag{2.20}$$

where S is a constant $N \times N$-matrix depending on the Kowalewski solution.

Solutions of (2.20) have the form $\boldsymbol{\Psi} = \mathbf{s}t^{-k}$, where k is an eigenvalue and $\mathbf{s}$ is an eigenvector of the matrix S. The number $1 - k$ is called *Kowalewski exponent*.

According to the Kowalewski–Lyapunov test, system (2.18) is 'integrable' if for any Kowalewski solution all corresponding Kowalewski exponents belong to an *a priori* fixed number set $\mathcal{A}$. The structure of $\mathcal{A}$ is closely related to analytic properties of the general solution for (2.18). The usual choice $\mathcal{A} = \mathbb{Z}$ is associated with the requirement that the general solution should be single valued. The latter is a standard assumption for the Painlevé analysis. The most general version $\mathcal{A} = \mathbb{Q}$ is associated with general solutions having algebraic branch points. But the main property for us is that $\mathcal{A}$ cannot be too wide. In particular, it cannot contain any open subset of $\mathbb{C}$ or $\mathbb{R}$. Therefore for any one-parameter family of integrable (in the Kowalewski–Lyapunov sense) dynamical systems (2.18) the Kowalewski exponents must not depend on the parameter. This gives us strong necessary integrability conditions for one-parametric families of homogeneous quadratic dynamical systems.

Given c_1 and c_2, the Hamiltonian (2.9) with (2.16) depends on one essential continuous parameter. Indeed, the length of $\mathbf{a}$ is fixed by (2.16), the length of $\mathbf{b}$ can be normalized by the scaling of the Hamiltonian. Two vectors of a fixed length have only one invariant (the angle between them) with respect to orthogonal transformation (1.5).

We want to find all pairs c_1 and c_2 in (2.9) such that the Hamiltonian is integrable for any value of the angle. Using transformation (1.5), we may reduce $\mathbf{b}$ and $\mathbf{a}$ to

$$\mathbf{b} = (0, 0, 1), \qquad \mathbf{a} = (a_1, 0, a_3). \tag{2.21}$$

Taking into account the constraint (2.16), we see that now a_3 remains to be the only free parameter in (2.9). For a generic Hamiltonian of this kind the Kowalewski exponents for the equations of motion depend continuously on a_3. For an integrable Hamiltonian these exponents must not depend on the parameter at all.

Theorem 1. *Suppose all Kowalewski exponents for Hamiltonian (2.9) with (2.16), (2.21) do not depend on a_3; then the pair of constants c_1, c_2 belongs (up to the transformation $c_1 \to -c_1, c_2 \to -c_2$, which corresponds to $\mathbf{a} \to -\mathbf{a}$) to the following list:*

(a) c_1—*arbitrary*, $c_2 = 0$,
(b) $c_1 = 1$, $c_2 = -2$,
(c) $c_1 = 1$, $c_2 = -1$,
(d) $c_1 = 1$, $c_2 = -\frac{1}{2}$,
(e) $c_1 = 1$, $c_2 = 1$.

To prove this statement, we have to investigate the Kowalewski exponents on all solutions of the form

$$M_i = \frac{m_i}{t}, \qquad \gamma_i = \frac{g_i}{t} \tag{2.22}$$

for the equations of motion defined by Hamiltonian (2.9), (2.16). To get the list of theorem 1, it turns out to be enough to investigate the Kowalewski exponents on special solutions (2.22), which satisfy $g_3 = 1$. There exist two classes of such solutions. For the first class we have $c_2(a_1m_2 - a_2m_1) = 1$. The second class is defined by $m_3 = 0$. For solutions of the first class, besides the cases of theorem 1, the only extra surviving possibility is $c_1 = 1/2, c_2 = -1$. But this case does not pass the Kowalewski–Lyapunov test on the solutions of the second class. For each pair (c_1, c_2) of the remaining list (a)–(e) of theorem 1 we verify that for all Kowalewski solutions the Kowalewski exponents do not depend on a_3. All computations are straightforward. A technical problem is that the explicit form of solutions of the first class involves radicals. To avoid this difficulty, we used calculations based on the Groebner basis technique.

Comments. *The Hamiltonian (a) belongs to the family*

$$H = c_1|\mathbf{b}|^2|\mathbf{M}|^2 + c_2(\mathbf{b}, \mathbf{M})^2 + (\mathbf{b}, \mathbf{M} \times \mathbf{\Gamma}), \tag{2.23}$$

which possesses (without restriction (2.16)) the linear integral of motion $I = (\mathbf{b}, \mathbf{M})$.

The Hamiltonians (b) and (c) are described in examples 2 and 1, correspondingly. The possibility (d) leads to the following new integrable case with an additional integral of sixth degree:

Example 3. The Hamiltonian,

$$H = (\mathbf{a}, \mathbf{b})|\mathbf{M}|^2 - \tfrac{1}{2}(\mathbf{a}, \mathbf{M})(\mathbf{b}, \mathbf{M}) + (\mathbf{b}, \mathbf{M} \times \mathbf{\Gamma}), \tag{2.24}$$

under condition (2.16) has the additional sixth degree integral

$$\begin{aligned} I = (\mathbf{b}, \mathbf{M})^2[&(\mathbf{b} \times \mathbf{a}, \mathbf{M} \times \mathbf{a})^2\mathbf{M}^2 + 2(\mathbf{b} \times \mathbf{a}, \mathbf{M} \times \mathbf{a})(\mathbf{b} \times \mathbf{a}, \mathbf{M} \times (\mathbf{M} \times \mathbf{\Gamma})) \\ &- \mathbf{\Gamma}^2(\mathbf{M}, \mathbf{b} \times \mathbf{a})^2 - (\mathbf{b} \times \mathbf{a}, \mathbf{M} \times \mathbf{\Gamma})^2 - \mathbf{M}^2\mathbf{\Gamma}^2(\mathbf{b}, \mathbf{a})^2 - \kappa\mathbf{M}^2\mathbf{\Gamma}^2\mathbf{b}^2]. \end{aligned}$$

Possibly this integrable case is related to the simple Lie algebra G_2.

In [17] a generalization of the general scheme by Sklyanin [16] has been proposed. In [15] the authors found a separation of variables for Hamiltonian (2.17) from example 2 in the framework of this approach. Probably a separation of variables for the Hamiltonians from examples 1 and 3 could be found after some development of these ideas.

All Kowalewski exponents for cases (a)–(c) are integers. In case (d) there are two solutions of class 2. For these solutions we have

$$\det(S - k\mathrm{Id}) = (k-1)^3(k+2)\left(k - \tfrac{1}{2}\right)\left(k + \tfrac{3}{2}\right)$$

and

$$\det(S - k\mathrm{Id}) = (k-1)^3(k+2)\left(k + \tfrac{1}{2}\right)^2.$$

In other words, some of the Kowalewski exponents are half-integers.

Case (e) is a mysterious one. We have verified that the Hamiltonian has no additional polynomial integrals of degrees less than or equal to 8. On the other hand, on all Kowalewski solutions all Kowalewski exponents are integers. It would be interesting to verify whether the equations of motion in case (e) satisfy the standard Painlevé test[4].

[4] Note added in proof: case (e) has been investigated by Sakovich who found that this case does not pass the Painlevé test (see [21]).

3. Inhomogeneous integrable Hamiltonians

3.1. Admissible linear terms for integrable homogeneous Hamiltonians

In this subsection we find for integrable homogeneous Hamiltonians H of the form (2.23), (2.17), (2.15), (2.13), (2.11) or (2.24) possible linear terms

$$T = (\mathbf{k}, \mathbf{M}) + (\mathbf{n}, \boldsymbol{\Gamma}), \tag{3.25}$$

where $\mathbf{k}$ and $\mathbf{n}$ are constant vectors, such that the Hamiltonian

$$\tilde{H} = H + T$$

has an additional integral of the same degree as H.

Proposition 1. *The following linear terms are admissible (in the above sense):*

(1)	*for Hamiltonian (2.23)*	$T = p_1(\mathbf{b}, \mathbf{M}) + p_2(\mathbf{b}, \boldsymbol{\Gamma})$,
(2)	*for Hamiltonian (2.17) with (2.16)*	$T = (\mathbf{k}, \mathbf{M}) + p_1(\mathbf{b}, \boldsymbol{\Gamma})$,
(3)	*for Hamiltonian (2.15) with (2.16)*	$T = (p_1\mathbf{a} + p_2\mathbf{a} \times \mathbf{b}, \mathbf{M}) + p_3(\mathbf{b}, \boldsymbol{\Gamma})$,
(4)	*for Hamiltonian (2.13)*	$T = (\mathbf{k}, \mathbf{M}) + p_1(\mathbf{v} \times \mathbf{z}, \boldsymbol{\Gamma})$,
(5)	*for Hamiltonian (2.11) with* $(\mathbf{u}, \mathbf{v}) = 0$	$T = p_1(\mathbf{u}, \mathbf{M}) + p_2(\mathbf{u} \times \mathbf{v}, \boldsymbol{\Gamma})$,
(6)	*for Hamiltonian (2.24) with (2.16)*	$T = p_1(\mathbf{a} \times \mathbf{b}, \mathbf{M})$,

where $\mathbf{k}$ *is an arbitrary vector and* p_1, p_2, p_3 *are arbitrary constants.*

We present the explicit form of the additional integrals for the non-homogeneous Hamiltonians $\tilde{H}$ of proposition 1 in the appendix.

3.2. A deformation of the Poincare model

The formula (2.23) describes Hamiltonians (1.3) that have linear additional integrals. We shall call (2.23) the Poincaré model.

For the special pair $c_1 = 1, c_2 = -1/2$ the Poincaré model is superintegrable. It means that besides the linear integral the Hamiltonian has a polynomial integral of degree higher than 1.

Proposition 2. *The Hamiltonian*

$$H_{\text{hom}} = |\mathbf{b}|^2|\mathbf{M}|^2 - \tfrac{1}{2}(\mathbf{b}, \mathbf{M})^2 + (\mathbf{b}, \mathbf{M} \times \boldsymbol{\Gamma}) \tag{3.26}$$

under condition $|\mathbf{b}|^2 = -\kappa$ *has the following additional integral of degree 4, functionally independent of H, the Casimirs and the linear integral* $(\mathbf{b}, \mathbf{M})$*:*

$$\begin{aligned} I = {} & (\mathbf{k}, \mathbf{M})[(\mathbf{k}, \mathbf{M})|\mathbf{b}|^2 - 2(\mathbf{k}, \mathbf{b})(\mathbf{b}, \mathbf{M})] \cdot [|\boldsymbol{\Gamma}|^2 + |\mathbf{b}|^2|\mathbf{M}|^2] \\ & + |\mathbf{M}|^2(\mathbf{b}, \mathbf{M})^2[(\mathbf{k}, \mathbf{b})^2 + |\mathbf{k}|^2|\mathbf{b}|^2] - (\mathbf{k} \times \mathbf{b}, \mathbf{M} \times \boldsymbol{\Gamma})^2 \\ & + 2(\mathbf{k} \times \mathbf{b}, \mathbf{b} \times \mathbf{M}) \cdot [|\mathbf{M}|^2(\mathbf{k}, \mathbf{b} \times \boldsymbol{\Gamma}) - (\mathbf{M}, \boldsymbol{\Gamma})(\mathbf{k}, \mathbf{b} \times \mathbf{M})], \end{aligned}$$

where $\mathbf{k}$ *is an arbitrary constant vector.*

It follows from the Jacobi identity that $I_1 = \{(\mathbf{b}, \mathbf{M}), I\}$ is a first integral as well. It turns out that $I_1 \neq 0$, which means that the Poisson subalgebra of polynomial integrals for H_{hom} is non-commutative. For any X let us denote the Poisson bracket $\{(\mathbf{b}, \mathbf{M}), X\}$ by X'. One can check that the integral I satisfies the relation $I''' + 4|\mathbf{b}|^2 I' = 0$.

The derivation $X \to X'$ can be regarded as a linear operator on the finite-dimensional vector space S_n of all homogeneous polynomials of degree n depending on components of $\mathbf{M}$ and $\boldsymbol{\Gamma}$. The operator spectrum is $\mu_k = \mathrm{i}k|\mathbf{b}|, 1 \leqslant k \leqslant n$. Probably this operator plays an important role in the theory of the vector Hamiltonians (2.9). Many terms in the integrable Hamiltonians from this class admit simple descriptions in terms of this operator. For example, the linear polynomial $(\mathbf{k}, \mathbf{b} \times M)$, where $\mathbf{k}$ is an arbitrary constant vector, is the general linear solution of the equation $X'' + |\mathbf{b}|^2 X = 0$. This fact and formula (2.10) imply that the arbitrary Hamiltonian (2.9) satisfies the equation $X''' + |\mathbf{b}|^2 X' = 0$. The additional integral from example 1 satisfies the same equation and so on.

The Hamiltonian (3.26) admits the following inhomogeneous integrable extension:

$$\tilde{H} = H_{\text{hom}} + (\mathbf{k} \times \mathbf{b}, \mathbf{M}) + p_1(\mathbf{b}, \mathbf{\Gamma}), \tag{3.27}$$

where $\mathbf{k}$ is an arbitrary constant vector, p_1 is an arbitrary constant. Hamiltonian (3.27) under condition (2.16) has an additional integral of degree 4, given in the appendix.

4. Classification results

It is very likely that all integrable Hamiltonians of the form (1.3) and (1.4) are exhausted by the examples presented in sections 2 and 3.

Theorem 2. *Suppose a Hamiltonian of the form (1.3) with real coefficients has an additional polynomial integral of degree from 1 to 8; then the Hamiltonian belongs to the six families (2.23), (2.17), (2.15), (2.13), (2.11) or (2.24).*

Scheme of the proof. For computations we use the canonical form (1.6). Without loss of generality we may assume that the additional polynomial integral is homogeneous. Given the degree m of the additional integral I, we form the general homogeneous mth degree polynomial of six variables M_i, γ_i with undetermined coefficients. The condition $\{I, H\} = 0$ gives rise to a bi-linear system of algebraic equations for both coefficients of H and I. Of course, this system can be solved 'by hand' only for small m. If $m = 6$ the algebraic system contains 791 bi-linear equations for 458 unknown coefficients. This system cannot be currently solved by standard computer algebra systems. Also all attempts by the authors to use the two best known packages specialized in the solution of polynomial algebraic systems failed.

The computation was performed using the computer algebra package CRACK. For $m < 6$ the calculations were performed automatically and for $m \geqslant 6$ with manual interaction.

Following the same line we have obtained. □

Theorem 3. *Suppose a Hamiltonian of the form (1.4) with real coefficients has an additional polynomial integral of degree from 1 to 6; then the Hamiltonian belongs to the seven families described in propositions 1 and 2.*

5. Conclusion: unsolved problems

This paper as well as [5] belongs mostly to so-called 'experimental' mathematical physics. The result of the experiment is a new interesting class (1.4) of quadratic Hamiltonians. This class contains several new integrable cases. Theorems 1–3 give reasons to believe that we found all real integrable Hamiltonians of the form (1.4). However, this should be proved. To do that, one can apply the Painlevé approach or methods developed in [20].

The separation of variables for several models from our paper is also an open problem.

Besides Hamiltonians listed in theorem 3, there exist integrable Hamiltonians of the form (1.4) with complex coefficients. For example, the Hamiltonian (2.9), (2.16) with $c_1 = 1, c_2 = -\frac{2}{3}$ has an additional integral of sixth degree under condition $|\mathbf{b} \times \mathbf{a}| = 0$. To find the complex Hamiltonians, one should consider two different normalizations of the vector **b**. The first is $\mathbf{b} = (0, 0, b_3)$, which corresponds to the possibility $|\mathbf{b}| \neq 0$ and gives rise to the normal form (1.6). The second normalization $\mathbf{b} = (0, b_2, ib_2)$ corresponds to $|\mathbf{b}| = 0$. In the case of fourth degree additional integrals all complex integrable Hamiltonians have been found in [5].

Probably all Hamiltonians from our paper have their quantum counterparts (for the case of fourth degree integral see [5]). It would be interesting to find the corresponding quantum Hamiltonians and integrals as well as the quantum separation of variables.

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Appendix

Here we present an explicit form of additional integrals for inhomogeneous Hamiltonians from propositions 1 and 2. Each integral I of degree m is a sum $I = \sum_{i=1}^{m} I_i$, where I_i is a polynomial of degree i homogeneous in $\mathbf{M}, \mathbf{\Gamma}$.

Case 1 of proposition 1. $I = I_1 = (\mathbf{b}, \mathbf{M})$.

Case 2 of proposition 1. I_3 is presented in example 2,

$$I_2 = 2p_1(\mathbf{b}, \mathbf{M})(\mathbf{a}, \mathbf{\Gamma}) - (\mathbf{k}, \mathbf{a})|\mathbf{M}|^2 - (\mathbf{k}, \mathbf{M} \times \mathbf{\Gamma}),$$
$$I_1 = -\kappa p_1^2(\mathbf{b}, \mathbf{M}) - p_1(\mathbf{k}, \mathbf{\Gamma}).$$

Case 3 of proposition 1. I_4 is presented in example 1,

$$\begin{aligned}
I_3 &= 2(\mathbf{b}, \mathbf{M})\{p_1[(\mathbf{a}, \mathbf{M})^2 + \kappa|\mathbf{M}|^2 - |\mathbf{\Gamma}|^2 - 2(\mathbf{a}, \mathbf{M} \times \mathbf{\Gamma})] \\
&\quad + p_2[(\mathbf{b} \times \mathbf{a}, \mathbf{M} \times \mathbf{\Gamma}) + (\mathbf{a}, \mathbf{M})(\mathbf{a}, \mathbf{b} \times \mathbf{M})] + p_3(\mathbf{b}, \mathbf{M})(\mathbf{a}, \mathbf{\Gamma})\}, \\
I_2 &= p_1^2[2|\mathbf{\Gamma}|^2 - (\mathbf{a}, \mathbf{M})^2 + 2(\mathbf{a}, \mathbf{M} \times \mathbf{\Gamma})] - 2p_2p_3(\mathbf{b}, \mathbf{M})(\mathbf{a}, \mathbf{b} \times \mathbf{\Gamma}) \\
&\quad + p_2^2[|\mathbf{b}|^2(\mathbf{a}, \mathbf{M})^2 - \kappa(\mathbf{b}, \mathbf{M})^2 - 2(\mathbf{a}, \mathbf{b})(\mathbf{a}, \mathbf{M})(\mathbf{b}, \mathbf{M})] - 4p_1p_3(\mathbf{b}, \mathbf{M})(\mathbf{a}, \mathbf{\Gamma}) \\
&\quad - p_3^2\kappa(\mathbf{b}, \mathbf{M})^2 + 2p_1p_2[(\mathbf{a} \times \mathbf{b}, \mathbf{M} \times \mathbf{\Gamma}) - (\mathbf{a}, \mathbf{M})(\mathbf{a}, \mathbf{b} \times \mathbf{M})] \\
I_1 &= 2p_1p_3[p_1(\mathbf{a}, \mathbf{\Gamma}) + p_2(\mathbf{a}, \mathbf{b} \times \mathbf{\Gamma}) + p_3\kappa(\mathbf{b}, \mathbf{M})].
\end{aligned}$$

Case 4 of proposition 1. $I_4 = \eta|\mathbf{a} \times \mathbf{b}|^2 I$ where I is given by (2.14),

$$\begin{aligned}
I_3 &= p_1\eta((\mathbf{v}, \mathbf{z})^2 - |\mathbf{v}|^2|\mathbf{z}|^2) \cdot [\eta|\mathbf{M}|^2(\mathbf{v}, \mathbf{z} \times \mathbf{\Gamma}) - 2(\mathbf{v}, \mathbf{\Gamma})(\mathbf{z}, \mathbf{M} \times \mathbf{\Gamma}) + |\mathbf{\Gamma}|^2(\mathbf{v}, \mathbf{z} \times \mathbf{M})] \\
&\quad + \eta^2|\mathbf{M}|^2[(\mathbf{z}, \mathbf{M})(\mathbf{v} \times \mathbf{z}, \mathbf{k} \times \mathbf{v}) - (\mathbf{v}, \mathbf{M})(\mathbf{v} \times \mathbf{z}, \mathbf{k} \times \mathbf{z})] \\
&\quad + 2\eta[(\mathbf{v}, \mathbf{M})(\mathbf{z}, \mathbf{k})(\mathbf{v} \times \mathbf{z}, \mathbf{\Gamma} \times \mathbf{M}) + (\mathbf{v}, \mathbf{M})(\mathbf{k}, \mathbf{M})(\mathbf{v} \times \mathbf{z}, \mathbf{z} \times \mathbf{\Gamma}) \\
&\quad - (\mathbf{v}, \mathbf{M})(\mathbf{k}, \mathbf{\Gamma})(\mathbf{v} \times \mathbf{z}, \mathbf{z} \times \mathbf{M}) \\
&\quad - \tfrac{1}{2}(\mathbf{k}, \mathbf{\Gamma})|\mathbf{M}|^2|\mathbf{v} \times \mathbf{z}|^2 - (\mathbf{z} \times \mathbf{M}, \mathbf{M} \times \mathbf{\Gamma})(\mathbf{v} \times \mathbf{z}, \mathbf{v} \times \mathbf{k})] \\
&\quad - (\mathbf{v}, \mathbf{z} \times \mathbf{k})(\mathbf{v}, \mathbf{z} \times \mathbf{M})|\mathbf{\Gamma}|^2, \\
I_2 &= p_1^2\eta|\mathbf{v} \times \mathbf{z}|^2[\eta^2(\mathbf{v} \times \mathbf{M}, \mathbf{z} \times \mathbf{M}) + (\mathbf{v}, \mathbf{\Gamma})(\mathbf{z}, \mathbf{\Gamma})] - p_1\eta[((\mathbf{v}, \mathbf{M})(\mathbf{z}, \mathbf{\Gamma}) \\
&\quad + (\mathbf{v}, \mathbf{\Gamma})(\mathbf{z}, \mathbf{M}))(\mathbf{v}, \mathbf{z} \times \mathbf{k}) + (\mathbf{v} \times \mathbf{z}, \mathbf{z} \times \mathbf{k})(\mathbf{v}, \mathbf{M} \times \mathbf{\Gamma}) \\
&\quad + (\mathbf{v} \times \mathbf{z}, \mathbf{v} \times \mathbf{k})(\mathbf{z}, \mathbf{M} \times \mathbf{\Gamma})] - \eta|\mathbf{M}|^2(\mathbf{v} \times \mathbf{k}, \mathbf{z} \times \mathbf{k}) + (\mathbf{v}, \mathbf{z} \times \mathbf{k})(\mathbf{k}, \mathbf{M} \times \mathbf{\Gamma}), \\
I_1 &= p_1(\mathbf{v}, \mathbf{z} \times \mathbf{k}) \cdot (p_1\eta^2(\mathbf{v}, \mathbf{z} \times \mathbf{M}) + (\mathbf{k}, \mathbf{\Gamma})).
\end{aligned}$$

Case 5 of proposition 1. I_4 is is given by (2.12), $I_1 = 0$,

$$I_3 = 2p_1(P|\mathbf{M}|^2 - (\mathbf{v},\mathbf{M})(\mathbf{M},\mathbf{\Gamma})) + 2p_2((\mathbf{M},\mathbf{\Gamma})(\mathbf{u},\mathbf{v}\times\mathbf{M}) - P(\mathbf{M},\Gamma\times\mathbf{v})),$$
$$I_2 = p_1^2|\mathbf{M}|^2 + p_2^2(|\mathbf{v}|^2|\mathbf{\Gamma}|^2 + \kappa(\mathbf{v},\mathbf{M})^2 - (\mathbf{v},\mathbf{\Gamma})^2) - 2p_1p_2(\mathbf{M},\Gamma\times\mathbf{v}),$$

where $P = (\mathbf{u},\mathbf{M}) + (\mathbf{v},\mathbf{\Gamma})$.

Case 6 of proposition 1. I_6 is presented in example 3, $I_1 = 0$,

$$\begin{aligned}
I_5 = 4p_1(\mathbf{b},\mathbf{M})\{&[(\mathbf{a},\mathbf{b})|\mathbf{M}|^2(\mathbf{a}\times\mathbf{b},\mathbf{a}\times\mathbf{M})(\mathbf{a},\mathbf{b}\times\mathbf{M}) + |\mathbf{\Gamma}|^2(\mathbf{a}\times\mathbf{b},\mathbf{b}\times\mathbf{M})(\mathbf{a},\mathbf{b}\times\mathbf{M})]\\
&+ (\mathbf{a}\times\mathbf{b},\mathbf{M}\times\mathbf{\Gamma})[(\mathbf{b},\mathbf{\Gamma})(\mathbf{a},\mathbf{b}\times\mathbf{M}) - (\mathbf{b},\mathbf{M})(\mathbf{a},\mathbf{b}\times\mathbf{\Gamma})]\\
&+ (\mathbf{b},\mathbf{M})(\mathbf{M},\mathbf{\Gamma})[(\mathbf{a},\mathbf{b})(\mathbf{a}\times\mathbf{b},\mathbf{a}\times\mathbf{M}) - \kappa(\mathbf{a}\times\mathbf{b},\mathbf{b}\times\mathbf{M})]\\
&- (\mathbf{a},\mathbf{b})(\mathbf{a}\times\mathbf{b},\mathbf{b}\times\mathbf{M})[2(\mathbf{a},\mathbf{M})(\mathbf{M},\mathbf{\Gamma}) - (\mathbf{a},\mathbf{\Gamma})|\mathbf{M}|^2]\\
&+ |\mathbf{M}|^2(\mathbf{a}\times\mathbf{b},\mathbf{a}\times\mathbf{M})[|\mathbf{b}|^2(\mathbf{a},\mathbf{\Gamma}) - 2(\mathbf{a},\mathbf{b})(\mathbf{b},\mathbf{\Gamma})]\},
\end{aligned}$$

$$\begin{aligned}
I_4 = p_1^2\{&4[(\mathbf{a}\times\mathbf{b},\mathbf{b}\times\mathbf{M})((\mathbf{a},\mathbf{\Gamma})(\mathbf{b},\mathbf{M}) + 2(\mathbf{a},\mathbf{b})(\mathbf{M},\mathbf{\Gamma})) - (\mathbf{b},\mathbf{\Gamma})(\mathbf{b},\mathbf{M})(\mathbf{a}\times\mathbf{b},\mathbf{a}\times\mathbf{M})\\
&- 2(\mathbf{a},\mathbf{b})|\mathbf{M}|^2(\mathbf{a}\times\mathbf{b},\mathbf{b}\times\mathbf{\Gamma})]\cdot(\mathbf{a},\mathbf{b}\times\mathbf{M}) + 4(\mathbf{b},\mathbf{M})[(\mathbf{b},\mathbf{M})(\mathbf{a}\times\mathbf{b},\mathbf{a}\times\mathbf{M})\\
&- (\mathbf{a},\mathbf{M})(\mathbf{a}\times\mathbf{b},\mathbf{b}\times\mathbf{M})]\cdot(\mathbf{a},\mathbf{b}\times\mathbf{\Gamma}) - 8(\mathbf{a}\times\mathbf{b},\mathbf{b}\times\mathbf{M})(\mathbf{a}\times\mathbf{b},\mathbf{b}\times\mathbf{\Gamma})(\mathbf{M},\mathbf{\Gamma})\\
&+ [4(\mathbf{a},\mathbf{b})^2(\mathbf{a},\mathbf{M})(\mathbf{a}\times\mathbf{b},\mathbf{b}\times\mathbf{M}) - 4(\mathbf{b},\mathbf{M})(2(\mathbf{a},\mathbf{b})^2 - |\mathbf{a}|^2|\mathbf{b}|^2)(\mathbf{a}\times\mathbf{b},\mathbf{a}\times\mathbf{M})\\
&+ 4(\mathbf{a}\times\mathbf{b},\mathbf{b}\times\mathbf{\Gamma})^2]|\mathbf{M}|^2 - 4(\mathbf{a},\mathbf{b})^2((\mathbf{a},\mathbf{b})^2 - |\mathbf{a}|^2|\mathbf{b}|^2)|\mathbf{M}|^4\},
\end{aligned}$$

$$I_3 = 8p_1^3|\mathbf{a}\times\mathbf{b}|^2((\mathbf{a},\mathbf{b})|\mathbf{M}|^2(\mathbf{a},\mathbf{b}\times\mathbf{M}) + (\mathbf{b}\times\mathbf{a},\mathbf{b}\times\mathbf{\Gamma})|\mathbf{M}|^2 + (\mathbf{a}\times\mathbf{b},\mathbf{b}\times\mathbf{M})(\mathbf{M},\mathbf{\Gamma})),$$
$$I_2 = -4p_1^4|\mathbf{a}\times\mathbf{b}|^2((\mathbf{a},\mathbf{b})^2|\mathbf{M}|^2 - |\mathbf{b}|^2|\mathbf{\Gamma}|^2).$$

The case of proposition 2. $I_1 = 0$; I_4 is related to integral I from proposition 2 by

$$\begin{aligned}
I_4 = |\mathbf{b}|^2 I + (\mathbf{b},\mathbf{M})^2&[(\mathbf{k}\times\mathbf{b},\mathbf{b}\times\mathbf{\Gamma})(\mathbf{k},\mathbf{b}\times\mathbf{M}) - (\mathbf{k}\times\mathbf{b},\mathbf{b}\times\mathbf{M})(\mathbf{k},\mathbf{b}\times\mathbf{\Gamma})\\
&+ ((\mathbf{k},\mathbf{b})^2 - |\mathbf{k}|^2|\mathbf{b}|^2)\cdot(|\mathbf{\Gamma}|^2 - \tfrac{1}{4}(\mathbf{b},\mathbf{M})^2) + |\mathbf{k}|^2|\mathbf{b}|^2(|\mathbf{\Gamma}|^2 + \kappa|\mathbf{M}|^2)],
\end{aligned}$$

$$\begin{aligned}
I_3 = p_1[&((\mathbf{k},\mathbf{b})^2 - |\mathbf{k}|^2|\mathbf{b}|^2)(\mathbf{b},\mathbf{M})^2(\mathbf{b},\mathbf{\Gamma}) - 2|\mathbf{b}|^2(\mathbf{k}\times\mathbf{b},\mathbf{M}\times\mathbf{\Gamma})((\mathbf{k},\mathbf{b}\times\mathbf{\Gamma})\\
&+ (\mathbf{k}\times\mathbf{b},\mathbf{b}\times\mathbf{M}))] + ((\mathbf{k},\mathbf{b})^2 - |\mathbf{k}|^2|\mathbf{b}|^2)(\mathbf{b},\mathbf{M})[(\mathbf{b},\mathbf{M})(\mathbf{k},\mathbf{b}\times\mathbf{M}) + 2(\mathbf{k}\times\mathbf{b},\mathbf{\Gamma}\times\mathbf{M})],
\end{aligned}$$

$$\begin{aligned}
I_2 = p_1^2[&|\mathbf{b}|^2(\mathbf{k}\times\mathbf{b},\mathbf{b}\times\mathbf{M})^2 + (\mathbf{k}\times\mathbf{b},\mathbf{b}\times\mathbf{\Gamma})^2 + ((\mathbf{k},\mathbf{b})^2 - |\mathbf{k}|^2|\mathbf{b}|^2)|\mathbf{b}\times\mathbf{\Gamma}|^2\\
&+ |\mathbf{k}|^2|\mathbf{b}|^4(|\mathbf{\Gamma}|^2 + \kappa|\mathbf{M}|^2)] - 2p_1((\mathbf{k},\mathbf{b})^2 - |\mathbf{k}|^2|\mathbf{b}|^2)(\mathbf{b},\mathbf{M})(\mathbf{k},\mathbf{b}\times\mathbf{\Gamma})\\
&+ ((\mathbf{k},\mathbf{b})^2 - |\mathbf{k}|^2|\mathbf{b}|^2)[(\mathbf{b},\mathbf{M})(\mathbf{k}\times\mathbf{b},\mathbf{k}\times\mathbf{M}) - (\mathbf{k},\mathbf{M})(\mathbf{k}\times\mathbf{b},\mathbf{b}\times\mathbf{M})].
\end{aligned}$$

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